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2010 J. Phys. A: Math. Theor. 43 045204

(http://iopscience.iop.org/1751-8121/43/4/045204)

J. Phys. A: Math. Theor. **43** (2010) 045204 (12pp) doi:10.1088/1751-8113/43/4/045204

Quasideterminant solutions of the generalized Heisenberg magnet model

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Received 29 September 2009, in final form 18 November 2009
Published 23 December 2009
Online at stacks.iop.org/JPhysA/43/045204

Abstract
In this paper we present the Darboux transformation for the generalized Heisenberg magnet (GHM) model based on the general linear Lie group $GL(n)$ and construct multi-soliton solutions in terms of quasideterminants. Further we relate the quasideterminant multi-soliton solutions obtained by means of Darboux transformation with those obtained by the dressing method. We also discuss the model based on the Lie group $SU(n)$ and obtain explicit soliton solutions of the model based on $SU(2)$.

PACS numbers: 11.10.Nx, 02.30.Ik

1. Introduction

During the past decades, there has been an increasing interest in the study of classical and quantum integrability of the Heisenberg ferromagnet (HM) model [1–15]. The Heisenberg ferromagnet (HM) model based on Hermitian symmetric spaces has been studied in [11–14]. The integrability of the HM model based on $SU(2)$ via the inverse scattering method is presented in [2, 3] and its $SU(n)$ generalization is studied in [4]. The integrability of a generalized HF (GHM) model based on the general linear Lie group $GL(n)$ via Lax formalism has been investigated in [1]. In this paper we present the Darboux transformation of the GHM model based on the general linear group $GL(n)$ with Lie algebra $\mathtt{gl(n)}$ and calculate multi-soliton solutions in terms of quasideterminants. We also establish the relation between the Darboux transformation and the well-known dressing method [16]. In the last section, we discuss the model-based $SU(n)$ and calculate an explicit expression of the single-soliton solution of the HM model based on the Lie group $SU(2)$ using Darboux transformation.

The Hamiltonian of the GHM model is defined by [1]

$$\mathcal{H} = \tfrac{1}{2}\,\mathrm{Tr}((\partial_x U)^T(\partial_x U)), \tag{1.1}$$

where 'T' is transpose and $U(x, t)$ is a matrix-valued function which takes values in the Lie algebra $\mathtt{gl(n)}$ of the general linear group $GL(n)$. The corresponding equation of motion can be expressed as

$$\partial_t U = \{\mathcal{H}, \partial_x U\}. \tag{1.2}$$

Equation (1.2) can be written as

$$\partial_t U = \left[U, \partial_x^2 U\right], \tag{1.3}$$

where $\partial_x = \frac{\partial}{\partial x}$ and $\partial_t = \frac{\partial}{\partial t}$. Let us assume that $U(x, t)$ is diagonizable, i.e.

$$U = gTg^{-1}, \tag{1.4}$$

where $g \in GL(n)$ is a matrix function of (x, t) and T is a $n \times n$ constant matrix

$$T = \begin{pmatrix} c_1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & c_1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & c_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & c_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & c_2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & c_2 \end{pmatrix}, \tag{1.5}$$

where $1 \leqslant p \leqslant n$ and $c_1, c_2 \in \mathbb{R}$ (or $\mathbb{C}$). From equations (1.4) and (1.5), we have

$$[U, [U, [U, \chi]]] = c^2 [U, \chi], \tag{1.6}$$

for an arbitrary matrix function χ and $c = c_1 - c_2 \neq 0$. Since

$$\partial_x U \equiv U_x = [\partial_x g g^{-1}, U], \tag{1.7}$$

it implies

$$[U, [U, U_x]] = c^2 U_x. \tag{1.8}$$

The equation of motion (1.3) can also be written as the zero-curvature condition, i.e.

$$\left[\partial_x - \frac{1}{(1-\lambda)}U, \partial_t - \frac{c^2}{(1-\lambda)^2}U - \frac{1}{(1-\lambda)}[U, U_x]\right] = 0. \tag{1.9}$$

The above zero-curvature condition (1.9) is equivalent to the compatibility condition of the following Lax pair:

$$\partial_x \Psi(x, t; \lambda) = \frac{1}{(1-\lambda)} U(x, t)\Psi(x, t; \lambda) \tag{1.10}$$

$$\partial_t \Psi(x, t; \lambda) = \left(\frac{c^2}{(1-\lambda)^2}U + \frac{1}{(1-\lambda)}[U, U_x]\right)\Psi(x, t; \lambda), \tag{1.11}$$

where λ is a real (or complex) parameter and Ψ is an invertible $n \times n$ matrix-valued function belonging to $GL(n)$.

In the next section, we define the Darboux transformation on matrix solutions Ψ of the Lax pair (1.10)–(1.11). To write down the explicit expressions for matrix solutions of the GHM model, we will use the notion of the quasideterminant introduced by Gelfand and Retakh [17–21].

Let X be an $n \times n$ matrix over a ring R (noncommutative, in general). For any $1 \leqslant i, j \leqslant n$, let r_i be the ith row and c_j be the jth column of X. There exist n^2 quasideterminants denoted by $|X|_{ij}$ for $i, j = 1, \ldots, n$ and are defined by

$$|X|_{ij} = \begin{vmatrix} X^{ij} & c_j^{\,i} \\ r_i^{\,j} & \boxed{x_{ij}} \end{vmatrix} = x_{ij} - r_i^j (X^{ij})^{-1} c_j^i, \tag{1.12}$$

where x_{ij} is the ijth entry of X, $r_i^{\,j}$ represents the ith row of X without the jth entry, $c_j^{\,i}$ represents the jth column of X without the ith entry and X^{ij} is the submatrix of X obtained by removing from X the ith row and the jth column. The quasideterminats are also denoted by the following notation. If the ring R is commutative, i.e. the entries of the matrix X all commute, then

$$|X|_{ij} = (-1)^{i+j} \frac{\det X}{\det X^{ij}}. \tag{1.13}$$

For a detailed account of quasideterminants and their properties see e.g. [17–21]. In this paper, we will consider only quasideterminants that are expanded about an $n \times n$ matrix over a commutative ring. Let

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1.14}$$

be a block decomposition of any $K \times K$ matrix where the matrix D is $n \times n$ and A is invertible. The ring R in this case is the (noncommutative) ring of $n \times n$ matrices over another commutative ring. The quasideterminant of $K \times K$ matrix expanded about the $n \times n$ matrix D is defined by

$$\begin{vmatrix} A & B \\ C & \boxed{D} \end{vmatrix} = D - CA^{-1}B. \tag{1.15}$$

The quasideterminants have found various applications in the theory of integrable systems, where the multisoliton solutions of various noncommutative integrable systems are expressed in terms of quasideterminants (see e.g. [22–30]).

2. Darboux transformation

The Darboux transformation is one of the well-known methods of obtaining multi-soliton solutions of many integrable models [31–33]. We define the Darboux transformation on the matrix solutions of the Lax pair (1.10)–(1.11), in terms of an $n \times n$ matrix $D(x, t, \lambda)$, called the Darboux matrix. For a general discussion on the Darboux matrix approach, see e.g. [34–39]. The Darboux matrix relates the two matrix solutions of the Lax pair (1.10)–(1.11) in such a way that the Lax pair is covariant under the Darboux transformation. The onefold Darboux transformation on the matrix solution of the Lax pair (1.10)–(1.11) is defined by

$$\Psi[1](x, t; \lambda) = D(x, t, \lambda)\Psi(x, t; \lambda), \tag{2.1}$$

where $D(x, t, \lambda)$ is the Darboux matrix. For our case, we can make the following ansatz

$$D(x, t, \lambda) = \lambda I - M(x, t), \tag{2.2}$$

where $M(x, t)$ is an $n \times n$ matrix function and I is an $n \times n$ identity matrix. The new solution $\Psi[1](x, t; \lambda)$ satisfies the following Lax pair, i.e.

$$\partial_x \Psi[1](x, t; \lambda) = \frac{1}{1-\lambda} U[1]\Psi[1](x, t; \lambda), \tag{2.3}$$

$$\partial_t \Psi[1](x,t;\lambda) = \left(\frac{c^2}{(1-\lambda)^2}U[1] + \frac{1}{1-\lambda}[U[1], U_x[1]]\right)\Psi[1](x,t;\lambda), \tag{2.4}$$

where $U[1]$ satisfies the equation of motion (1.3). By operating ∂_x and ∂_t on equation (2.1) and equating the coefficients of different powers of λ, we get the following transformation on the matrix field U

$$U[1] = U + M_x, \tag{2.5}$$

and the following conditions which M is required to satisfy

$$M_x(I-M) = [U, M], \tag{2.6}$$

$$M_t(I-M)^2 = [c^2U + [U, U_x], M] + M[U, U_x]M - [U, U_x]M^2. \tag{2.7}$$

One can solve equations (2.6–2.7) to obtain an explicit expression for the matrix function $M(x,t)$. An explicit expression for $M(x,t)$ can be found as follows.

Let us take n distinct real (or complex) constant parameters $\lambda_1, \ldots, \lambda_n$ $(\neq 1)$. Also take n constant column vectors $e_1, e_2, \ldots, e_n$ and construct an invertible non-degenerate $n\times n$ matrix function $\Theta(x,t)$

$$\Theta(x,t) = (\Psi(\lambda_1)e_1, \ldots, \Psi(\lambda_n)e_n) = (\theta_1, \ldots, \theta_n). \tag{2.8}$$

Each column $\theta_i = \Psi(\lambda_i)e_i$ in the matrix Θ is a column solution of the Lax pair (1.10)–(1.11) when $\lambda = \lambda_i$ and $i = 1, 2, \ldots, n$, i.e.

$$\partial_x\theta_i = \frac{1}{1-\lambda_i}U\theta_i, \tag{2.9}$$

$$\partial_t\theta_i = \left(\frac{c^2}{(1-\lambda_i)^2}U + \frac{1}{1-\lambda_i}[U, U_x]\right)\theta_i. \tag{2.10}$$

Let us take an $n\times n$ invertible diagonal matrix with entries being eigenvalues λ_i corresponding to the eigenvectors θ_i

$$\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n). \tag{2.11}$$

The $n \times n$ matrix generalization of the Lax pair (2.9)–(2.10) will be

$$\partial_x\Theta = U\Theta(I-\Lambda)^{-1}, \tag{2.12}$$

$$\partial_t\Theta_i = c^2U\Theta(I-\Lambda)^{-2} + [U, U_x]\Theta(I-\Lambda)^{-1}. \tag{2.13}$$

The $n \times n$ matrix Θ is a particular matrix solution of the Lax pair (2.9)–(2.10) with Λ being a matrix of particular eigenvalues. In terms of a particular matrix solution Θ of the Lax pair (2.9)–(2.10), we make the following choice of the matrix $M(x,t)$:

$$M(x,t) = \Theta\Lambda\Theta^{-1}. \tag{2.14}$$

Our next step is to check that equation (2.14) is a solution of equations (2.6)–(2.7). In order to show this, we first operate ∂_x on equation (2.14) to get

$$\begin{aligned}\partial_x M &= \partial_x(\Theta\Lambda\Theta^{-1}),\\ &= (\partial_x\Theta)\Lambda\Theta^{-1} + \Theta\Lambda\partial_x(\Theta^{-1}),\\ &= U\Theta(I-\Lambda)^{-1}\Lambda\Theta^{-1} - \Theta\Lambda\Theta^{-1}U\Theta(I-\Lambda)^{-1}\Theta^{-1},\\ &= -U + \Theta(I-\Lambda)\Theta^{-1}j_+\Theta(I-\Lambda)^{-1}\Theta^{-1},\\ &= -U + (I-M)U(I-M)^{-1},\end{aligned} \tag{2.15}$$

which is equation (2.6). Similarly when we operate ∂_t on (2.14), we get

$$\begin{aligned}\partial_t M &= \partial_t(\Theta\Lambda\Theta^{-1})\\ &= (\partial_t\Theta)\Lambda\Theta^{-1} + \Theta\Theta\Lambda\partial_t(\Theta^{-1})\\ &= (c^2 U\Theta(I-\Lambda)^{-2} + [U, U_x]\Theta(I-\Lambda)^{-1})\Lambda\Theta^{-1}\\ &\quad - \Theta\Lambda\Theta^{-1}(c^2 U\Theta(I-\Lambda)^{-2} + [U, U_x]\Theta(I-\Lambda)^{-1})\Theta^{-1},\end{aligned} \tag{2.16}$$

which is equation (2.7). This shows that the choice (2.14) of the matrix M satisfies equations (2.6)–(2.7). In other words we can say that if the collection (Ψ, U) is a solution of the Lax pair (1.10)–(1.11) and the matrix M is defined by (2.14), then $(\Psi[1], U[1])$ defined by (2.1) and (2.5), respectively, is also a solution of the same Lax pair. Therefore, we say that

$$\begin{aligned}\Psi[1] &= (\lambda I - \Theta\Lambda\Theta^{-1})\Psi,\\ U[1] &= (I - \Theta\Lambda\Theta^{-1})U(I - \Theta\Lambda\Theta^{-1})^{-1}\end{aligned}$$

is the required Darboux transformation on the solution Ψ to the Lax pair (1.10)–(1.11) and U to the equation of motion (1.3), respectively.

3. Quasideterminant solutions

We have shown that the matrix $M = \Theta\Lambda\Theta^{-1}$ satisfies the conditions (2.6)–(2.7). Therefore, the onefold Darboux transformation (2.1) can also be written in terms of quasideterminants as

$$\begin{aligned}\Psi[1] \equiv D(x,t;\lambda)\Psi &= \left(\lambda I - \Theta_1\Lambda_1\Theta_1^{-1}\right)\Psi,\\ &= \begin{vmatrix}\Theta_1 & \Psi\\ \Theta_1\Lambda_1 & \boxed{\lambda\Psi}\end{vmatrix}.\end{aligned} \tag{3.1}$$

The above equation defines the Darboux transformation on the matrix solution Ψ of the Lax pair (1.10)–(1.11). The corresponding onefold Darboux transformation on the matrix field U is

$$\begin{aligned}U[1] &= \left(I - \Theta_1\Lambda_1\Theta_1^{-1}\right)U\left(I - \Theta_1\Lambda_1\Theta_1^{-1}\right)^{-1},\\ &= \begin{vmatrix}\Theta_1 & I\\ \Theta_1(I-\Lambda_1) & \boxed{0}\end{vmatrix} U \begin{vmatrix}\Theta_1 & I\\ \Theta_1(I-\Lambda_1) & \boxed{0}\end{vmatrix}^{-1}.\end{aligned} \tag{3.2}$$

We write the twofold Darboux transformation on Ψ as

$$\begin{aligned}\Psi[2] \equiv D(x,t;\lambda)\Psi[1] &= \lambda\Psi[1] - \Theta_2[1]\Lambda_2\Theta_2^{-1}[1]\Psi[1]\\ &= \lambda\left(\lambda I - \Theta_1\Lambda_1\Theta_1^{-1}\right)\Psi\\ &\quad - \left(\Theta_2\Lambda_2 - \Theta_1\Lambda_1\Theta_1^{-1}\Theta_2\right)\Lambda_2\left(\Theta_2\Lambda_2 - \Theta_1\Lambda_1\Theta_1^{-1}\Theta_2\right)^{-1}\left(\lambda I - \Theta_1\Lambda_1\Theta_1^{-1}\right)\Psi,\\ &= \begin{vmatrix}\Theta_1 & \Theta_2 & \Psi\\ \Theta_1\Lambda_1 & \Theta_2\Lambda_2 & \lambda\Psi\\ \Theta_1\Lambda_1^2 & \Theta_2\Lambda_2^2 & \boxed{\lambda^2\Psi}\end{vmatrix}.\end{aligned} \tag{3.3}$$

Similarly the expression for the twofold Darboux transformation on the matrix field U is

$$\begin{aligned}U[2] &= \Theta_2[1](I-\Lambda_2)\Theta_2^{-1}[1]U[1]\left(\Theta_2[1](I-\Lambda_2)\Theta_2^{-1}[1]\right)^{-1},\\ &= \left(\Theta_2\Lambda_2 - \Theta_1\Lambda_1\Theta_1^{-1}\Theta_2\right)(I-\Lambda_2)\left(\Theta_2\Lambda_2 - \Theta_1\Lambda_1\Theta_1^{-1}\Theta_2\right)^{-1}\\ &\quad \times\left(I - \Theta_1\Lambda_1\Theta_1^{-1}\right)U\left(I - \Theta_1\Lambda_1\Theta_1^{-1}\right)^{-1}\\ &\quad \times\left(\left(\Theta_2\Lambda_2 - \Theta_1\Lambda_1\Theta_1^{-1}\Theta_2\right)(I-\Lambda_2)\left(\Theta_2\Lambda_2 - \Theta_1\Lambda_1\Theta_1^{-1}\Theta_2\right)^{-1}\right)^{-1},\end{aligned}$$

$$= \begin{vmatrix} \Theta_1 & \Theta_2 & I \\ \Theta_1 (I-\Lambda_1) & \Theta_2 (I-\Lambda_2) & 0 \\ \Theta_1 (I-\Lambda_1)^2 & \Theta_2 (I-\Lambda_2)^2 & \boxed{0} \end{vmatrix} \times U \times \begin{vmatrix} \Theta_1 & \Theta_2 & I \\ \Theta_1 (I-\Lambda_1) & \Theta_2 (I-\Lambda_2) & 0 \\ \Theta_1 (I-\Lambda_1)^2 & \Theta_2 (I-\Lambda_2)^2 & \boxed{0} \end{vmatrix}^{-1}. \tag{3.4}$$

The result can be generalized to obtain N-fold Darboux transformation on matrix solution Ψ as

$$\Psi[N] = \begin{vmatrix} \Theta_1 & \Theta_2 & \cdots & \Theta_N & \Psi \\ \Theta_1\Lambda_1 & \Theta_2\Lambda_2 & \cdots & \Theta_N\Lambda_N & \lambda\Psi \\ \Theta_1\Lambda_1^2 & \Theta_2\Lambda_2^2 & \cdots & \Theta_N\Lambda_N^2 & \lambda^2\Psi \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta_1\Lambda_1^N & \Theta_2\Lambda_2^N & \cdots & \Theta_N\Lambda_N^N & \boxed{\lambda^N\Psi} \end{vmatrix}. \tag{3.5}$$

Similarly the expression for $U[N]$ is

$$U[N] = \begin{vmatrix} \Theta_1 & \Theta_2 & \cdots & \Theta_N & I \\ \Theta_1 (I-\Lambda_1) & \Theta_2 (I-\Lambda_2) & \cdots & \Theta_N (I-\Lambda_N) & 0 \\ \Theta_1 (I-\Lambda_1)^2 & \Theta_2 (I-\Lambda_2)^2 & \cdots & \Theta_N (I-\Lambda_N)^2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta_1 (I-\Lambda_1)^N & \Theta_2 (I-\Lambda_2)^N & \cdots & \Theta_N (I-\Lambda_N)^N & \boxed{0} \end{vmatrix} \times U \times \begin{vmatrix} \Theta_1 & \Theta_2 & \cdots & \Theta_N & I \\ \Theta_1 (I-\Lambda_1) & \Theta_2 (I-\Lambda_2) & \cdots & \Theta_N (I-\Lambda_N) & 0 \\ \Theta_1 (I-\Lambda_1)^2 & \Theta_2 (I-\Lambda_2)^2 & \cdots & \Theta_N (I-\Lambda_N)^2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta_1 (I-\Lambda_1)^N & \Theta_2 (I-\Lambda_2)^N & \cdots & \Theta_N (I-\Lambda_N)^N & \boxed{0} \end{vmatrix}^{-1}. \tag{3.6}$$

We now relate the quasideterminant solutions of GHM with the solutions obtained by the dressing method and the inverse scattering method. For this purpose, we proceed as follows. From the definition of the matrix M, we have

$$M\Theta = \Theta\Lambda. \tag{3.7}$$

Let θ_i and θ_j be the column solutions of the Lax pair (1.10)–(1.11) when $\lambda = \lambda_i$ and $\lambda = \lambda_j$, respectively, i.e.

$$\begin{aligned} M\theta_i &= \lambda_i\theta_i, \qquad i = 1, 2, \ldots, p \\ M\theta_j &= \lambda_j\theta_j, \qquad j = p+1, \quad p+2, \ldots, n. \end{aligned} \tag{3.8}$$

Now we take $\lambda_i = \mu$ and $\lambda_j = \bar{\mu}$, and we may write the matrix M as

$$M = \mu P + \bar{\mu} P^{\perp}, \tag{3.9}$$

where P is the Hermitian projector i.e. $P^{\dagger} = P$. The projector P satisfies $P^2 = P$ and $P^{\perp} = 1 - P$. The projector P is the Hermitian projection on a complex space and $P^{\perp}$ is the projection on an orthogonal space. Now equation (3.9) can also written as

$$M = (\mu - \bar{\mu})\, P + \bar{\mu} I, \tag{3.10}$$

where the Hermitian projector can be expressed as

$$P = \theta_i \left(\theta_i^{\dagger}, \theta_i\right)^{-1} \theta_i^{\dagger}. \tag{3.11}$$

The onefold Darboux transformation (3.1) on the matrix solution Ψ can also be expressed in terms of projector P as

$$\Psi[1] \equiv \mathcal{D}(x,t;\lambda)\Psi = \left(I - \frac{\mu - \bar{\mu}}{\lambda - \bar{\mu}} P\right)\Psi, \tag{3.12}$$

where $\mathcal{D}(x,t;\lambda)$ is the rescaled Darboux-dressing function, i.e. $\mathcal{D}(x,t;\lambda) = (\lambda - \mu)^{-1} D(x,t;\lambda)$. Similarly the N-fold Darboux transformation (3.5) on the matrix solution Ψ can also be written as (take $P[1] = P$)

$$\Psi[N] = \prod_{k=0}^{N-1}\left(I - \frac{\mu_{N-k} - \bar{\mu}_{N-k}}{\lambda - \bar{\mu}_{N-k}} P[N-k]\right)\Psi. \tag{3.13}$$

Now we can express the N-fold Darboux transformation (3.6) on the matrix field U that can be written as

$$U[N] = \prod_{k=0}^{N-1}\left(I - \frac{\mu_{N-k} - \bar{\mu}_{N-k}}{1 - \bar{\mu}_{N-k}} P[N-k]\right) U \prod_{l=1}^{N-1}\left(I - \frac{\bar{\mu}_l - \mu_l}{1 - \bar{\mu}_l} P[l]\right), \tag{3.14}$$

and the Hermitian projector is defined as

$$P[k] = \theta_i[k]\big(\theta_i^{\dagger}[k], \theta_i[k]\big)^{-1}\theta_i^{\dagger}[k]. \tag{3.15}$$

Expressions (3.13) and (3.14) can also be written as the sum of K terms [27]:

$$\Psi[N] = \sum_{k=0}^{N-1}\left(I - \frac{1}{\lambda - \bar{\mu}_k} R_k\right)\Psi, \tag{3.16}$$

and

$$U[N] = \sum_{k=0}^{N-1}\left(I - \frac{1}{1 - \bar{\mu}_k} R_k\right) U \sum_{l=0}^{N-1}\left(I - \frac{1}{1 - \bar{\mu}_l} R_l\right)^{-1}, \tag{3.17}$$

where

$$R_k = \sum_{l=0}^{N-1}(\mu_l - \bar{\mu}_k)\theta_i^{(k)}\big(\theta_i^{(k)\dagger}, \theta_i^{(l)}\big)^{-1}\theta_i^{(l)\dagger}. \tag{3.18}$$

4. The explicit solutions of the GHM model

In this section, we calculate explicit expression of the soliton solution. First of all we will study the GHM model based on $SU(n)$. In this case, the spin function U takes values in the Lie algebra $\mathtt{su(n)}$ so that one can decompose the spin function into components $U = U^a T^a$, and T^a, $a = 1, 2, \ldots, n^2$ are anti-Hermitian $n \times n$ matrices with normalization $\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$ and are the generators of the $SU(n)$ in the fundamental representation satisfying the algebra

$$[T^a, T^b] = f^{abc} T^c, \tag{4.1}$$

where f^{abc} are the structure constants of the Lie algebra $\mathtt{su(n)}$. For any $X \in \mathtt{su(n)}$, we write $X = X^a T^a$ and $U^a = -2\,\mathrm{Tr}(U T^a)$.

The matrix-field U belongs to the Lie algebra $\mathtt{su(n)}$ of the Lie group $SU(n)$; therefore,

$$U^{\dagger} = -U, \qquad \mathrm{Tr}(U) = 0. \tag{4.2}$$

Equations (2.1)–(2.2) and (2.5) define a Darboux transformation for the GHM model based on the Lie group $SU(n)$. The new solution of the equation of motion (1.3) $U[1]$ must be $\mathtt{su(n)}$ valued, i.e.

$$U^{\dagger}[1] = -U[1], \qquad \mathrm{Tr}(U[1]) = 0; \tag{4.3}$$

therefore, we have the following conditions on the matrix M:

$$M^{\dagger} = -M, \qquad \mathrm{Tr}(M) = 0. \tag{4.4}$$

In other words we want to make specific M to satisfy the (4.4). This can be achieved if we choose the particular solutions θ_i at $\lambda = \lambda_i$; let us first calculate

$$\begin{aligned}\partial_x\left(\theta_i^{\dagger}\theta_j\right) &= \left(\partial_x\theta_i^{\dagger}\right)\theta_j + \theta_i^{\dagger}\left(\partial_x\theta_j\right)\\ &= (1-\bar{\lambda}_i)^{-1}\theta_i^{\dagger}U^{\dagger}\theta_j + (1-\lambda_j)^{-1}\theta_i^{\dagger}U\theta_j;\end{aligned} \tag{4.5}$$

using equation (4.2) equation (4.5) becomes

$$\partial_x\left(\theta_i^{\dagger}\theta_j\right) = 0, \tag{4.6}$$

when $\lambda_i \neq \lambda_j$ (i.e. $\bar{\lambda}_i = \lambda_j$). Similarly we can check

$$\partial_t\left(\theta_i^{\dagger}\theta_j\right) = 0. \tag{4.7}$$

From the definition of the matrix M, we have

$$\theta_i^{\dagger}(M^{\dagger} + M)\theta_j = (\bar{\lambda}_i + \lambda_j)\theta_i^{\dagger}\theta_j, \tag{4.8}$$

when $\lambda_i \neq \lambda_j$ and then expression (4.8) implies

$$\theta_i^{\dagger}\theta_j = 0. \tag{4.9}$$

The column vectors θ_i are linearly independent and equation (4.9) holds everywhere.

For the HM model based on $SU(n)$, the constant matrix (1.5) becomes

$$T = \begin{pmatrix} 2-\frac{2}{n} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0\\ 0 & -\frac{2}{n} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & -\frac{2}{n} & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & -\frac{2}{n} & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 0 & -\frac{2}{n} & \cdots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -\frac{2}{n} \end{pmatrix}. \tag{4.10}$$

Then U^2 becomes

$$U^2 = \frac{4(n-1)}{n^2}I + \frac{2(n-2)}{n}U. \tag{4.11}$$

These are the constraints given in [4]. For the construction of explicit soliton solution for the $SU(n)$ HM model, we construct the matrix M by defining a Hermitian projector P. For this case, we take the seed solution to be

$$U_0 \equiv U = \mathrm{i}\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}, \tag{4.12}$$

where a_i are real constants and $\sum_{i=1}^{n} a_i = 0$. The corresponding solution of the Lax pair is expressed in the block diagonal matrix:

$$\Psi(x,t;\lambda) = \begin{pmatrix} W_p(\lambda) & O \\ O & W_{n-p}(\lambda) \end{pmatrix}, \tag{4.13}$$

where

$$W_p(\lambda) = \begin{pmatrix} e^{i\omega_1(\lambda)} & & \\ & \ddots & \\ & & e^{i\omega_p(\lambda)} \end{pmatrix} \tag{4.14}$$

and

$$W_{n-p}(\lambda) = \begin{pmatrix} e^{i\omega_{p+1}(\lambda)} & & \\ & \ddots & \\ & & e^{i\omega_n(\lambda)} \end{pmatrix} \tag{4.15}$$

are $p \times p$ and $(n-p) \times (n-p)$ matrices, respectively, and

$$\omega_i(\lambda) = a_i \left(\frac{1}{1-\lambda} x + \frac{4}{(1-\lambda)^2} t \right). \tag{4.16}$$

Now define a particular matrix solution Θ of the Lax pair as

$$\Theta = (\Psi(\mu)L_1, \Psi(\bar{\mu})L_2), \tag{4.17}$$

where L_1 is an $n \times p$ constant matrix of p column vectors and L_2 is the orthogonal complementary $n \times (n-p)$ matrix of $(n-p)$ column vectors. The columns of L_1 span a p-dimensional subspace U of C^n, and those of L_2 span the orthogonal subspace V. The projector P is completely characterized by the two subspaces $U = \operatorname{Im} P$ and $V = \operatorname{Ker} P$ given by the condition $P^{\perp}U = 0$ and $PV = 0$. Let us write $L_1 = \binom{A}{B}$ and $L_2 = \binom{C}{D}$, where A, B, C and D are constant $p \times p$, $(n-p) \times n$, $p \times (n-p)$ and $(n-p) \times (n-p)$ constant matrices, respectively. Given this, the $n \times n$ matric Θ is given by

$$\Theta = \begin{pmatrix} W_p(\mu)A & W_p(\bar{\mu})C \\ W_{n-p}(\mu)B & W_{n-p}(\bar{\mu})D \end{pmatrix}. \tag{4.18}$$

We now define the projector P in terms of the matrix $\Phi = \Psi(\mu)L_1 = (\theta_1, \ldots, \theta_p)$ given by

$$\Phi = (\theta_1, \ldots, \theta_p) = \begin{pmatrix} W_p(\mu)A \\ W_{n-p}(\mu)B \end{pmatrix}.$$

The projector is thus given by

$$P = \begin{pmatrix} W_p(\mu)A\Delta A^{\dagger}W_p^{\dagger}(\bar{\mu}) & W_p(\mu)A\Delta B^{\dagger}W_{n-p}^{\dagger}(\bar{\mu}) \\ W_{n-p}(\mu)B\Delta A^{\dagger}W_p^{\dagger}(\bar{\mu}) & W_{n-p}(\mu)B\Delta B^{\dagger}W_{n-p}^{\dagger}(\bar{\mu}) \end{pmatrix}, \tag{4.19}$$

where $\Delta^{-1} = A^{\dagger}W_p^{\dagger}(\bar{\mu})W_p(\mu)A + B^{\dagger}W_{n-p}^{\dagger}(\bar{\mu})W_{n-p}(\mu)A$. The Darboux matrix $D(\lambda)$ can now be constructed to give an explicit soliton solution of the $SU(n)$ HM model. To elaborate the result more explicitly, we proceed with the example of the $SU(2)$ HM model.

For the $SU(2)$ model, equations (4.10) and (4.11) become

$$T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.20}$$

Then U^2 becomes

$$U^2 = I. \tag{4.21}$$

The Lax pair (1.10)–(1.11) for the $SU(2)$ model can be written as

$$\partial_x \Psi(x,t;\lambda) = \frac{1}{(1-\lambda)} U(x,t)\Psi(x,t;\lambda), \tag{4.22}$$

$$\partial_t \Psi(x,t;\lambda) = \left(\frac{4}{(1-\lambda)^2} U + \frac{2}{(1-\lambda)} U U_x \right) \Psi(x,t;\lambda). \tag{4.23}$$

If we take a trivial solution (as seed solution), single soliton and multi-soliton solutions can be obtained by Darboux transformation as explained above.

We take the seed solution to be

$$U_0 \equiv U = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}. \tag{4.24}$$

The corresponding solution of the linear system (4.22)–(4.23) can be written as

$$\Psi(x,t;\lambda) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\left(\frac{1}{(1-\lambda)}x + \frac{4}{(1-\lambda)^2}t\right)} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\left(\frac{1}{(1-\lambda)}x + \frac{4}{(1-\lambda)^2}t\right)} \end{pmatrix}. \tag{4.25}$$

Taking $\lambda_1 = \mu$ and $\lambda_2 = \bar{\mu}$, the constant matrix Λ is given by

$$\Lambda = \begin{pmatrix} \mu & 0 \\ 0 & \bar{\mu} \end{pmatrix}, \tag{4.26}$$

and the corresponding 2×2 matrix solution Θ becomes

$$\Theta \equiv (\theta_1, \theta_2) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\left(\frac{1}{(1-\mu)}x + \frac{4}{(1-\mu)^2}t\right)} & \mathrm{e}^{\mathrm{i}\left(\frac{1}{(1-\bar{\mu})}x + \frac{4}{(1-\bar{\mu})^2}t\right)} \\ -\mathrm{e}^{-\mathrm{i}\left(\frac{1}{(1-\mu)}x + \frac{4}{(1-\mu)^2}t\right)} & \mathrm{e}^{-\mathrm{i}\left(\frac{1}{(1-\bar{\mu})}x + \frac{4}{(1-\bar{\mu})^2}t\right)} \end{pmatrix}. \tag{4.27}$$

The matrix M is given by

$$\begin{aligned} M &= \Theta \Lambda \Theta^{-1}, \\ &= \frac{1}{\mathrm{e}^{u} + \mathrm{e}^{-u}} \begin{pmatrix} \mu\,\mathrm{e}^{u} + \bar{\mu}\,\mathrm{e}^{-u} & (\bar{\mu}-\mu)\,\mathrm{e}^{\mathrm{i}v} \\ (\bar{\mu}-\mu)\,\mathrm{e}^{-\mathrm{i}v} & \bar{\mu}\,\mathrm{e}^{u} + \mu\,\mathrm{e}^{-u} \end{pmatrix}, \end{aligned} \tag{4.28}$$

where the functions $u(x,t)$ and $v(x,t)$ are defined by

$$\begin{aligned} u(x,t) &= \mathrm{i}\left(\frac{1}{(1-\mu)} - \frac{1}{(1-\bar{\mu})} \right) x + 4\mathrm{i} \left(\frac{1}{(1-\mu)^2} - \frac{1}{(1-\bar{\mu})^2} \right) t, \\ v(x,t) &= \left(\frac{1}{(1-\mu)} + \frac{1}{(1-\bar{\mu})} \right) x + 4 \left(\frac{1}{(1-\mu)^2} + \frac{1}{(1-\bar{\mu})^2} \right) t. \end{aligned} \tag{4.29}$$

Let us take the eigenvalue to be $\mu = \mathrm{e}^{\mathrm{i}\theta}$. Expression (4.28) then becomes

$$M = \begin{pmatrix} \cos\theta + \mathrm{i}\sin\theta\tanh u & -\mathrm{i}\,(\sin\theta\,\mathrm{sech}\,u)\;\mathrm{e}^{\mathrm{i}v} \\ -\mathrm{i}\,(\sin\theta\,\mathrm{sech}\,u)\;\mathrm{e}^{-\mathrm{i}v} & \cos\theta - \mathrm{i}\sin\theta\tanh u \end{pmatrix}, \tag{4.30}$$

and the corresponding Darboux matrix $D(\lambda)$ in this case is

$$D(\lambda) = \begin{pmatrix} \lambda - \cos\theta - \mathrm{i}\sin\theta\tanh u & \mathrm{i}(\sin\theta\,\mathrm{sech}\,u)\,\mathrm{e}^{\mathrm{i}v} \\ \mathrm{i}(\sin\theta\,\mathrm{sech}\,u)\,\mathrm{e}^{-\mathrm{i}v} & \lambda - \cos\theta + \mathrm{i}\sin\theta\tanh u \end{pmatrix}. \tag{4.31}$$

Comparing the above equation with (3.12), we find the following expression for the projector

$$P = \begin{pmatrix} 2\,\mathrm{e}^{u}\mathrm{sech}\,u & -2\,\mathrm{e}^{\mathrm{i}v}\mathrm{sech}\,u \\ -2\,\mathrm{e}^{-\mathrm{i}v}\mathrm{sech}\,u & 2\,\mathrm{e}^{-u}\mathrm{sech}\,u \end{pmatrix}. \tag{4.32}$$

Using (3.2) and (4.24), we get

$$U[1] = \begin{pmatrix} \mathrm{i}U_3 & U_+ \\ -U_- & -\mathrm{i}U_3 \end{pmatrix}, \tag{4.33}$$

where

$$\begin{aligned} U_3 &= 1 - (1 + \cos\theta)\,\mathrm{sech}^2 u, \\ U_+ &\equiv \overline{U}_- = -\mathrm{i}\,\mathrm{e}^{\mathrm{i}v}\,[(1 + \cos\theta)\tanh u + \mathrm{i}\sin\theta]\,\mathrm{sech}\,u. \end{aligned} \tag{4.34}$$

From equation (4.34), we see that $U^{\dagger}[1] = -U[1]$ and $\mathrm{Tr}(U[1]) = 0$. Therefore, equation (4.34) is an explicit expression of the single-soliton solution of the HM model based on $SU(2)$ obtained by using Darboux transformation. Similarly one can calculate an explicit expression for the multi-soliton solution of the model. Expression (4.34) is similar to the expression of the single soliton given in [2].

5. Concluding remarks

In this paper, we have studied the GHM model based on the general linear Lie group $GL(n)$ and expressed the multi-soliton solutions in terms of the quasideterminant using the Darboux transformation defined on the solution of the Lax pair. We have also established equivalence between the Darboux matrix approach and Zakharov–Mikhailov's dressing method. In the last section, we have reduced the GHM model into the HM model based on $SU(n)$ and calculated an explicit expression for the single-soliton solution. It would be interesting to study the GHM models based on Hermitian symmetric spaces. We shall address this problem in a separate work.

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